SUPPLEMENT TO "MATRIX COMPLETION METHODS FOR THE TOTAL ELECTRON CONTENT VIDEO RECONSTRUCTION"

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A. Proof of Theoretical Results in Section 2.3.

A.1. *Proof of Theorem 2.1.*

PROOF. The objective function $F(A_{1:T}^{(k)}, B_{1:T}^{(k)})$ has the property:

$$\begin{aligned} F(A_{1:T}^{(k)}, B_{1:T}^{(k)}) &= \tilde{Q}(A_1^{(k)}|A_{1:T}^{(k)}, B_{1:T}^{(k)}) \\ &\geq \inf_{A_1} \tilde{Q}(A_1|A_{1:T}^{(k)}, B_{1:T}^{(k)}) \\ &= \tilde{Q}(A_1^{(k+1)}|A_{1:T}^{(k)}, B_{1:T}^{(k)}) \qquad (1) \\ &\geq F(A_1^{(k+1)}, A_{2:T}^{(k)}, B_{1:T}^{(k)}), \qquad (2) \end{aligned}$$

where the definition of $\tilde{Q}$ is in (9) of the paper. Equation (1) holds because we update A_1 to be $A_1^{(k+1)}$ using ridge regression: $A_1^{(k+1)} = \arg\min \tilde{Q}(A_1|A_{1:T}^{(k)}, B_{1:T}^{(k)})$. Inequality (2) holds because $\tilde{Q}(A_1^{(k+1)}|A_{1:T}^{(k)}, B_{1:T}^{(k)})$ is the upper bound of $F(A_1^{(k+1)}, A_{2:T}^{(k)}, B_{1:T}^{(k)})$, as we majorize the first term of the objective function using inequality (6) of the paper.

The property above indicates that after one single update of matrix A_1, the values of the objective function is non-increasing. Applying a similar argument for all other matrices $A_2, A_3, \dots, A_T, B_1, B_2, \dots, B_T$ leads to a chain of inequalities:

$$\begin{aligned} F(A_{1:T}^{(k)}, B_{1:T}^{(k)}) &\geq F(A_1^{(k+1)}, A_{2:T}^{(k)}, B_{1:T}^{(k)}) \geq F(A_{1:2}^{(k+1)}, A_{3:T}^{(k)}, B_{1:T}^{(k)}) \geq \cdots \geq F(A_{1:T}^{(k+1)}, B_{1:T}^{(k)}) \\ &\geq F(A_{1:T}^{(k+1)}, B_1^{(k)}, B_{2:T}^{(k)}) \geq F(A_{1:T}^{(k+1)}, B_{1:2}^{(k)}, B_{3:T}^{(k)}) \geq \dots F(A_{1:T}^{(k+1)}, B_{1:T}^{(k+1)}), \end{aligned}$$

which proves that the each update of A_t or B_t goes towards a descent direction. □

A.2. *Proof of Theorem 2.2.*

PROOF. Note that in appendix A.1, we proved inequality (2). More generally, for any arbitrary t, we have the following:

$$\begin{aligned} \Delta_{k,t}^{A} \triangleq & F(A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)}) - F(A_{1:t}^{(k+1)}, A_{t+1:T}^{(k)}, B_{1:T}^{(k)}) \qquad (3) \\ \geq & \tilde{Q}(A_t^{(k)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)}) - \tilde{Q}(A_t^{(k+1)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)}). \end{aligned}$$

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The right hand side of (3) is the difference of $\tilde{Q}(A_t|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)})$ evaluated at $A_t^{(k)}$ and $A_t^{(k+1)}$. Recall that:

$$\tilde{Q}(A_t|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)}) \triangleq \frac{1}{2}\|X_t^{(k)} - A_t(B_t^{(k)})^T\|_F^2 + \frac{\lambda_1}{2}\|A_t\|_F^2 + \frac{\lambda_3}{2}\|Y_t - A_t(B_t^{(k)})^T\|_F^2$$
$$+ \frac{\lambda_2}{2}\mathbf{I}_{\{t>1\}}\|A_t(B_t^{(k)})^T - A_{t-1}^{(k+1)}(B_{t-1}^{(k)})^T\|_F^2$$
$$+ \frac{\lambda_2}{2}\mathbf{I}_{\{t<T\}}\|A_{t+1}^{(k)}(B_{t+1}^{(k)})^T - A_t(B_t^{(k)})^T\|_F^2.$$

Note that this is a quadratic function of A_t thus higher order (≥ 3) derivatives are all zero. We can do a Taylor expansion for $\tilde{Q}(A_t^{(k)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)})$ at $A_t^{(k+1)}$:

$$\tilde{Q}(A_t^{(k)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)}) = \tilde{Q}(A_t^{(k+1)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)})$$
$$+ (\nabla\tilde{Q})(A_t^{(k)} - A_t^{(k+1)})$$
$$+ \frac{1}{2}(A_t^{(k)} - A_t^{(k+1)})^T H (A_t^{(k)} - A_t^{(k+1)}), \tag{4}$$

where $H = (1 + \lambda_2(1 + \mathbf{I}_{\{2\leq t\leq T-1\}}) + \lambda_3)(B_t^{(k)})^T B_t^{(k)} + \lambda_1 I$. We have $\nabla\tilde{Q} = 0$ since $A_t^{(k+1)}$ is the minimizer of $\tilde{Q}(A_t|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)})$. Combining (3) and (4), one can see that:

$$\Delta_{k,t}^A \geq \tilde{Q}(A_t^{(k)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)}) - \tilde{Q}(A_t^{(k+1)}|A_{1:t-1}^{(k+1)}, A_{t:T}^{(k)}, B_{1:T}^{(k)})$$
$$= \frac{1}{2}(A_t^{(k)} - A_t^{(k+1)})^T H (A_t^{(k)} - A_t^{(k+1)})$$
$$= \frac{1 + \lambda_2(1 + \mathbf{I}_{\{2\leq t\leq T-1\}}) + \lambda_3}{2}\|(A_t^{(k)} - A_t^{(k+1)})(B_t^{(k)})^T\|^2$$
$$+ \frac{\lambda_1}{2}\|A_t^{(k)} - A_t^{(k+1)}\|^2. \tag{5}$$

Similarly for any updates of B_t, we have:

$$\Delta_{k,t}^B \triangleq F(A_{1:T}^{(k+1)}, B_{1:t-1}^{(k+1)}, B_{t:T}^{(k)}) - F(A_{1:T}^{(k+1)}, B_{1:t}^{(k+1)}, B_{t+1:T}^{(k)})$$
$$\geq \tilde{Q}(B_t^{(k)}|A_{1:T}^{(k+1)}, B_{1:t-1}^{(k+1)}, B_{t:T}^{(k)}) - \tilde{Q}(B_t^{(k+1)}|A_{1:T}^{(k+1)}, B_{1:t-1}^{(k+1)}, B_{t:T}^{(k)})$$
$$= \frac{1 + \lambda_2(1 + \mathbf{I}_{\{2\leq t\leq T-1\}}) + \lambda_3}{2}\|A_t^{(k+1)}(B_t^{(k)} - B_t^{(k+1)})^T\|^2$$
$$+ \frac{\lambda_1}{2}\|B_t^{(k)} - B_t^{(k+1)}\|^2. \tag{6}$$

Since (5) and (6) hold for all A_t and B_t, we can sum the $\Delta_{k,t}^A, \Delta_{k,t}^B$ across all t. Note that $\sum_t(\Delta_{k,t}^A + \Delta_{k,t}^B) = \Delta_k$. The right-hand side is the lower bound for Δ_k that we want. □

A.3. *Proof of Theorem 2.3.*

PROOF. The first result can be easily proved by noting that

$$F(A_{1:T}^{(1)}, B_{1:T}^{(1)}) - f^\infty \geq F(A_{1:T}^{(1)}, B_{1:T}^{(1)}) - F(A_{1:T}^{(K)}, B_{1:T}^{(K)}) = \sum_{k=1}^K \Delta_k \geq K(\min_{1\leq k\leq K}\Delta_k). \tag{7}$$

Given the assumption that $l^L \mathrm{I} \leq (A_t^{(k)})^T A_t^{(k)} \leq l^U \mathrm{I}$, $l^L \mathrm{I} \leq (B_t^{(k)})^T B_t^{(k)} \leq l^U \mathrm{I}$ for all t, k. Equations (18) and (19) of the paper can be proved with the following inequalities:

$$l^L \|A_t^{(k)} - A_t^{(k+1)}\|^2 \leq \|(A_t^{(k)} - A_t^{(k+1)})(B_t^{(k)})^T\|^2 \leq l^U \|A_t^{(k)} - A_t^{(k+1)}\|^2; \tag{8}$$

$$l^L \|B_t^{(k)} - B_t^{(k+1)}\|^2 \leq \|A_t^{(k+1)}(B_t^{(k)} - B_t^{(k+1)})^T\|^2 \leq l^U \|B_t^{(k)} - B_t^{(k+1)}\|^2. \tag{9}$$

Given the lower bound in theorem 2.2 and the inequality in (7), we have:

$$\begin{aligned}
&\frac{F(A_{1:T}^{(1)}, B_{1:T}^{(1)}) - f^\infty}{K} \geq \min_{1 \leq k \leq K} \Delta_k \\
&\geq \min_{1 \leq k \leq K} \left\{ \frac{\lambda_1}{2} \sum_{t=1}^{T} \left(\|A_t^{(k)} - A_t^{(k+1)}\|^2 + \|B_t^{(k)} - B_t^{(k+1)}\|^2 \right) \right. \\
&\left. + \frac{1}{2} \sum_{t=1}^{T} (1 + \lambda_2 + \lambda_3) \left(\|(A_t^{(k)} - A_t^{(k+1)})(B_t^{(k)})^T\|^2 + \|A_t^{(k+1)}(B_t^{(k)} - B_t^{(k+1)})^T\|^2 \right) \right\} \\
&\geq \min_{1 \leq k \leq K} \left\{ \frac{l^L(1 + \lambda_2 + \lambda_3) + \lambda_1}{2} \sum_{t=1}^{T} \left(\|A_t^{(k)} - A_t^{(k+1)}\|^2 + \|B_t^{(k)} - B_t^{(k+1)}\|^2 \right) \right\}.
\end{aligned}$$

The last step uses the left inequality in (8) and (9). This proves (18). Using the right-hand side inequality in (8) and (9) yields (19). □